

# Thermodynamics of cosmological horizons in $f(T)$ gravity

Kazuharu Bamba<sup>1,2a</sup> and Chao-Qiang Geng<sup>1,3b</sup>

<sup>1</sup>*Department of Physics, National Tsing Hua University, Hsinchu, Taiwan 300*

<sup>2</sup>*Kobayashi-Maskawa Institute for the Origin of Particles and the Universe,  
Nagoya University, Nagoya 464-8602, Japan*

<sup>3</sup>*Physics Division, National Center for Theoretical Sciences, Hsinchu, Taiwan 300*

## Abstract

We explore thermodynamics of the apparent horizon in  $f(T)$  gravity with both equilibrium and non-equilibrium descriptions. We find the same dual equilibrium/non-equilibrium formulation for  $f(T)$  as for  $f(R)$  gravity. In particular, we show that the second law of thermodynamics can be satisfied for the universe with the same temperature outside and inside the apparent horizon.

PACS numbers: 04.50.Kd, 04.70.Dy, 95.36.+x, 98.80.-k

---

<sup>a</sup> E-mail address: bamba@kmi.nagoya-u.ac.jp

<sup>b</sup> E-mail address: geng@phys.nthu.edu.tw

## I. INTRODUCTION

The current accelerating expansion of the universe has been supported by many cosmological observations such as Type Ia Supernovae [1], cosmic microwave background (CMB) radiation [2, 3], large scale structure [4], baryon acoustic oscillations [5], and weak lensing [6]. There are two representative approaches to explain the late time acceleration of the universe: One is to introduce some unknown matters called “dark energy” in the framework of general relativity (for a review on dark energy, see, e.g., [7, 8]). The other is to modify the gravitational theory, e.g.,  $f(R)$  gravity [9–12].

To explore gravity beyond general relativity, “teleparallelism” could be considered by using the Weitzenböck connection, which has no curvature but torsion, rather than the curvature defined by the Levi-Civita connection (see, e.g., [13]). This approach was also taken by Einstein [14]. The teleparallel Lagrangian density described by the torsion scalar  $T$  has been promoted to a function of  $T$ , *i.e.*,  $f(T)$ , in order to account for the late time cosmic acceleration [15, 16] as well as inflation [17]. This concept is similar to the idea of  $f(R)$  gravity. Various aspects of  $f(T)$  gravity have been examined in the literature [18–21]. In particular, the presence of extra degrees of freedom and the violation of local Lorentz invariance as well as the existence of non-trivial frames for  $f(T)$  gravity have been noted [21]. Clearly, more studies on  $f(T)$  gravity are needed to see if the theory is a viable one. Recently, the first law of thermodynamics in  $f(T)$  gravity has been studied in Ref. [22]<sup>1</sup>. In this paper, we concentrate on thermodynamics in  $f(T)$  gravity.

Black hole thermodynamics [23] indicated the fundamental connection between gravitation and thermodynamics (for recent reviews, see [24]). In Ref. [26], the Einstein equation was derived from the Clausius relation in thermodynamics with the proportionality of the entropy to the horizon area in general relativity. The procedure in Ref. [26] was developed to more general extended gravitational theories [27, 28]. On the other hand, to derive the corresponding gravitational field equation by using the formulation in Ref. [26] in  $f(R)$  gravity, a non-equilibrium thermodynamic treatment was proposed [29]. It has been recently demonstrated that one can acquire an equilibrium description of thermodynamics on the apparent horizon in the Friedmann-Lemaître-Robertson-Walker (FLRW) space-time for modified gravity theories with the Lagrangian density  $f(R, \phi, X)$ , where  $X = -(1/2) g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi$  is

<sup>1</sup> We note that the procedure for the discussions on the first law of thermodynamics in  $f(T)$  gravity used in Ref. [22] is different from that in this paper.

the kinetic term of a scalar field  $\phi$  ( $\nabla_\mu$  is the covariant derivative operator associated with the metric tensor  $g_{\mu\nu}$ ), by redefining an energy momentum tensor of “dark” components so that a local energy conservation law can be verified [30]. The same consequence has been obtained in  $f(R)$  gravity in the Palatini formalism [31].

In this paper, we investigate both non-equilibrium and equilibrium descriptions of thermodynamics in  $f(T)$  gravity. We also derive the conditions required by the second law of thermodynamics. We use units of  $k_B = c = \hbar = 1$  and denote the gravitational constant  $8\pi G$  by  $\kappa^2 \equiv 8\pi/M_{\text{Pl}}^2$  with the Planck mass of  $M_{\text{Pl}} = G^{-1/2} = 1.2 \times 10^{19} \text{GeV}$ .

The paper is organized as follows. In Sec. II, we explain  $f(T)$  gravity. In Sec. III, we investigate the non-equilibrium description of thermodynamics and explore the first and second laws of thermodynamics of the apparent horizon. In Sec. IV, we study the equilibrium description of thermodynamics. Finally, conclusions are given in Sec. V.

## II. $f(T)$ GRAVITY

In the teleparallelism, one uses orthonormal tetrad components  $e_A(x^\mu)$ , where an index  $A$  runs over  $0, 1, 2, 3$  for the tangent space at each point  $x^\mu$  of the manifold. Their relation to the metric  $g^{\mu\nu}$  is given by

$$g_{\mu\nu} = \eta_{AB} e_\mu^A e_\nu^B, \quad (2.1)$$

where  $\mu$  and  $\nu$  are coordinate indices on the manifold and also run over  $0, 1, 2, 3$ , and  $e_A^\mu$  forms the tangent vector of the manifold.

The torsion  $T^\rho_{\mu\nu}$  and contorsion  $K^{\mu\nu}{}_\rho$  tensors are defined by

$$T^\rho_{\mu\nu} \equiv e_A^\rho (\partial_\mu e_\nu^A - \partial_\nu e_\mu^A), \quad (2.2)$$

$$K^{\mu\nu}{}_\rho \equiv -\frac{1}{2} (T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}), \quad (2.3)$$

respectively. Instead of the Ricci scalar  $R$  for the Lagrangian density in general relativity, the teleparallel Lagrangian density is described by the torsion scalar  $T$ , defined as

$$T \equiv S_\rho{}^{\mu\nu} T^\rho_{\mu\nu}, \quad (2.4)$$

where

$$S_\rho{}^{\mu\nu} \equiv \frac{1}{2} (K^{\mu\nu}{}_\rho + \delta_\rho^\mu T^{\alpha\nu}{}_\alpha - \delta_\rho^\nu T^{\alpha\mu}{}_\alpha). \quad (2.5)$$

The modified teleparallel action for  $f(T)$  gravity is given by [16]

$$I = \int d^4x |e| \left[ \frac{f(T)}{2\kappa^2} + \mathcal{L}_M \right], \quad (2.6)$$

where  $|e| = \det(e_\mu^A) = \sqrt{-g}$  and  $\mathcal{L}_M$  is the matter Lagrangian. Varying the action in Eq. (2.6) with respect to the vierbein vector field  $e_A^\mu$ , we obtain [15]

$$\frac{1}{e} \partial_\mu (e S_A^{\mu\nu}) f' - e_A^\lambda T_{\mu\lambda}^\rho S_\rho^{\nu\mu} f' + S_A^{\mu\nu} \partial_\mu (T) f'' + \frac{1}{4} e_A^\nu f = \frac{\kappa^2}{2} e_A^\rho T_{\rho}^{(M)\nu}, \quad (2.7)$$

where  $T_{\rho}^{(M)\nu}$  is the contribution to the energy-momentum tensor from all perfect fluids of ordinary matter (radiation and non-relativistic matter).

We assume the four-dimensional flat Friedmann-Lemaître-Robertson-Walker (FLRW) space-time with the metric,

$$ds^2 = h_{\alpha\beta} dx^\alpha dx^\beta + \tilde{r}^2 d\Omega^2, \quad (2.8)$$

where  $\tilde{r} = a(t)r$ ,  $x^0 = t$  and  $x^1 = r$  with the two-dimensional metric  $h_{\alpha\beta} = \text{diag}(1, -a^2(t))$ . Here,  $a(t)$  is the scale factor and  $d\Omega^2$  is the metric of two-dimensional sphere with unit radius. In this space-time,  $g_{\mu\nu} = \text{diag}(1, -a^2, -a^2, -a^2)$  and the tetrad components  $e_\mu^A = (1, a, a, a)$  yield the exact value of torsion scalar  $T = -6H^2$ , where  $H = \dot{a}/a$  is the Hubble parameter and the dot denotes the time derivative of  $\partial/\partial t$ .

In the flat FLRW background, it follows from Eq. (2.7) that the modified Friedmann equations are given by [15, 16]

$$H^2 = \frac{1}{6F} \left( \kappa^2 \rho_M - \frac{f}{2} \right), \quad (2.9)$$

$$\dot{H} = -\frac{1}{4TF' + 2F} \left( \kappa^2 P_M - TF + \frac{f}{2} \right), \quad (2.10)$$

where  $F \equiv df/dT$ ,  $F' = dF/dT$ , and  $\rho_M$  and  $P_M$  are the energy density and pressure of all perfect fluids of generic matter, respectively. We note that the perfect fluid satisfies the continuity equation

$$\dot{\rho}_M + 3H(\rho_M + P_M) = 0. \quad (2.11)$$

Before we proceed to the next section, we would summarize our motivation of this paper. First,  $f(T)$  gravity can realize the accelerated expansion of the universe (not only the late time cosmic acceleration but also inflation). Second,  $f(T)$  gravity has the second-order gravitational field equation in derivatives, similar to that in general relativity, whereas the

gravitational field equation of  $f(R)$  gravity is fourth-order in derivatives. Hence, it is important to investigate the theoretical aspects in order to examine whether  $f(T)$  gravity can be an alternative gravitational theory to general relativity. In this paper, we concentrate on the first and second laws of thermodynamics of  $f(T)$  gravity.

### III. NON-EQUILIBRIUM DESCRIPTION OF THERMODYNAMICS IN $f(T)$ GRAVITY

#### A. Energy density and pressure of dark components

Equations (2.9) and (2.10) can be rewritten as

$$H^2 = \frac{\kappa^2}{3F} (\hat{\rho}_{\text{DE}} + \rho_{\text{M}}), \quad (3.1)$$

$$\dot{H} = -\frac{\kappa^2}{2F} (\hat{\rho}_{\text{DE}} + \hat{P}_{\text{DE}} + \rho_{\text{M}} + P_{\text{M}}), \quad (3.2)$$

where  $\hat{\rho}_{\text{DE}}$  and  $\hat{P}_{\text{DE}}$  are the energy density and pressure of “dark” components, given by

$$\hat{\rho}_{\text{DE}} \equiv \frac{1}{2\kappa^2} (FT - f), \quad (3.3)$$

$$\hat{P}_{\text{DE}} \equiv \frac{1}{2\kappa^2} [-(FT - f) + 4H\dot{F}], \quad (3.4)$$

leading to

$$\dot{\hat{\rho}}_{\text{DE}} + 3H(\hat{\rho}_{\text{DE}} + \hat{P}_{\text{DE}}) = -\frac{T}{2\kappa^2} \dot{F}. \quad (3.5)$$

Here, a hat denotes quantities in the non-equilibrium description of thermodynamics as the standard continuity equation does not hold due to  $\dot{F} \neq 0$  in Eq. (3.5).

#### B. First law of thermodynamics

We investigate the thermodynamic property of  $f(T)$  gravity. The relation  $h^{\alpha\beta} \partial_\alpha \tilde{r} \partial_\beta \tilde{r} = 0$  determines the dynamical apparent horizon<sup>2</sup>. In the flat FLRW space-time, the radius  $\tilde{r}_A$  of the apparent horizon is given by

$$\tilde{r}_A = \frac{1}{H}. \quad (3.6)$$

---

<sup>2</sup> In Refs. [32, 33], by using the recent type Ia Supernovae data it has been shown that the accelerating universe enveloped by the apparent horizon satisfies the generalized second law of thermodynamics, whereas the accelerating universe inside the event horizon does not. Clearly, from the thermodynamics point of view, the enveloping surface should be the apparent horizon and not the event one in the accelerating universe.

The time derivative of this relation gives to

$$-\frac{d\tilde{r}_A}{\tilde{r}_A^3} = \dot{H}H dt. \quad (3.7)$$

Substituting Eq. (3.2) into (3.7), we obtain

$$\frac{F}{4\pi G} d\tilde{r}_A = \tilde{r}_A^3 H \left( \hat{\rho}_t + \hat{P}_t \right) dt, \quad (3.8)$$

where  $\hat{\rho}_t \equiv \hat{\rho}_{\text{DE}} + \rho_{\text{M}}$  and  $\hat{P}_t \equiv \hat{P}_{\text{DE}} + P_{\text{M}}$  are the total energy density and pressure of the universe, respectively.

Note that in general relativity, the Bekenstein-Hawking horizon entropy is described by  $S = A/(4G)$ , where  $A = 4\pi\tilde{r}_A^2$  is the area of the apparent horizon [23]. In modified gravity theories including  $f(R)$  gravity, a horizon entropy  $\hat{S}$  associated with a Noether charge, called the Wald entropy  $\hat{S}$  [34], is expressed as  $\hat{S} = A/(4G_{\text{eff}})$ , where  $G_{\text{eff}} = G/f'$  with  $f' = df(R)/dR$  is the effective gravitational coupling in  $f(R)$  gravity [35]. We remark that the Wald entropy in  $f(R)$  gravity in both the metric [34, 36] and Palatini [37] formalisms take the same form.

According to the study of the matter density perturbations in  $f(T)$  gravity, the effective gravitational coupling in  $f(T)$  gravity is given by  $G_{\text{eff}} = G/F$  [20], similar to that in  $f(R)$  gravity. Furthermore, recently, in Ref. [22], by using the Wald's Noether charge method [34] and the related insights acquired in Refs. [26, 28, 29, 38], it has been shown that when  $F' = dF(T)/dT = d^2f(T)/dT^2$  is small, the entropy of black holes in  $f(T)$  gravity is approximately equal to  $FA/(4G)$ . Hence, we take the Wald entropy in  $f(T)$  gravity as

$$\hat{S} = \frac{FA}{4G}. \quad (3.9)$$

It is known that  $f(T)$  gravity is not local Lorentz invariant [21], which indicates the existence of new degrees of freedom. However, at the background level no new degrees of freedom are present, while at linear perturbation the new vector degree of freedom only satisfies constraint equations [21]. It seems that these new degrees of freedom should not directly contribute to physical observables, especially in the early universe [39]. Accordingly, we will assume that they do not contribute to the entropy in our study.

Using Eqs. (3.8) and (3.9), we find

$$\frac{1}{2\pi\tilde{r}_A} d\hat{S} = 4\pi\tilde{r}_A^3 H \left( \hat{\rho}_t + \hat{P}_t \right) dt + \frac{\tilde{r}_A}{2G} dF. \quad (3.10)$$

The associated temperature of the apparent horizon has the following Hawking temperature  $T_H$

$$T_H = \frac{|\kappa_{\text{sg}}|}{2\pi}, \quad (3.11)$$

with  $\kappa_{\text{sg}}$  being the surface gravity, given by [40]

$$\kappa_{\text{sg}} = \frac{1}{2\sqrt{-h}} \partial_\alpha \left( \sqrt{-h} h^{\alpha\beta} \partial_\beta \tilde{r} \right) \quad (3.12)$$

$$= -\frac{1}{\tilde{r}_A} \left( 1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A} \right) = -\frac{\tilde{r}_A}{2} \left( 2H^2 + \dot{H} \right) = -\frac{2\pi G}{3F} \tilde{r}_A \left( \hat{\rho}_t - 3\hat{P}_t \right), \quad (3.13)$$

where  $h$  is the determinant of the metric  $h_{\alpha\beta}$ . From Eq. (3.13), we see that  $\kappa_{\text{sg}} \leq 0$  if the total equation of state (EoS)  $w_t \equiv \hat{P}_t/\hat{\rho}_t$  satisfies  $w_t \leq 1/3$ . Using Eqs. (3.11) and (3.13), we have

$$T_H = \frac{1}{2\pi\tilde{r}_A} \left( 1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A} \right). \quad (3.14)$$

By multiplying the term  $1 - \dot{\tilde{r}}_A/(2H\tilde{r}_A)$  for Eq. (3.10), we acquire

$$T_H d\hat{S} = 4\pi\tilde{r}_A^3 H \left( \hat{\rho}_t + \hat{P}_t \right) dt - 2\pi\tilde{r}_A^2 \left( \hat{\rho}_t + \hat{P}_t \right) d\tilde{r}_A + \frac{T_H}{G} \pi\tilde{r}_A^2 dF. \quad (3.15)$$

In general relativity, the Misner-Sharp energy [41] is defined as  $E \equiv \tilde{r}_A/(2G)$ . Since  $G_{\text{eff}} = G/F$  [20] in  $f(T)$  gravity, this may be extended to

$$\hat{E} = \frac{\tilde{r}_A F}{2G}, \quad (3.16)$$

similar to that in  $f(R)$  gravity [42–44] (for related works, see also [45]). By combining Eqs. (3.6) and (3.16), we find

$$\hat{E} = V \frac{3FH^2}{8\pi G} = V\hat{\rho}_t, \quad (3.17)$$

where  $V = 4\pi\tilde{r}_A^3/3$  is the volume inside the apparent horizon. The last equality in Eq. (3.17) means that  $\hat{E}$  corresponds to the total intrinsic energy. It is clear from Eq. (3.17) that  $F > 0$  so that  $\hat{E} > 0$ . In this case, the effective gravitational coupling  $G_{\text{eff}} = G/F$  in  $f(T)$  gravity becomes positive like  $f(R)$  gravity [10]. Note that this condition is required to ensure that the graviton is not a ghost in the sense of quantum theory [46].

Using Eqs. (2.11) and (3.5), we find

$$d\hat{E} = -4\pi\tilde{r}_A^3 H \left( \hat{\rho}_t + \hat{P}_t \right) dt + 4\pi\tilde{r}_A^2 \hat{\rho}_t d\tilde{r}_A + \frac{\tilde{r}_A}{2G} dF. \quad (3.18)$$

It follows from Eqs. (3.15) and (3.18) that

$$T_H d\hat{S} = d\hat{E} + 2\pi\tilde{r}_A^2 \left( \hat{\rho}_d + \rho_f - \hat{P}_d - P_f \right) d\tilde{r}_A + \frac{\tilde{r}_A}{2G} (1 + 2\pi\tilde{r}_A T_H) dF. \quad (3.19)$$

By introducing the work density [47]

$$\hat{W} \equiv -\frac{1}{2} \left( T^{(\text{M})\alpha\beta} h_{\alpha\beta} + \hat{T}^{(\text{DE})\alpha\beta} h_{\alpha\beta} \right) \quad (3.20)$$

$$= \frac{1}{2} \left( \hat{\rho}_{\text{t}} - \hat{P}_{\text{t}} \right) \quad (3.21)$$

with  $\hat{T}^{(\text{DE})\alpha\beta}$  being the energy-momentum tensor of dark components, Eq. (3.19) is rewritten to

$$T_{\text{H}} d\hat{S} = -d\hat{E} + \hat{W} dV + \frac{\tilde{r}_A}{2G} (1 + 2\pi\tilde{r}_A T_{\text{H}}) dF, \quad (3.22)$$

which can be described as

$$T_{\text{H}} d\hat{S} + T_{\text{H}} d_i \hat{S} = -d\hat{E} + \hat{W} dV, \quad (3.23)$$

where

$$d_i \hat{S} = -\frac{1}{T_{\text{H}}} \frac{\tilde{r}_A}{2G} (1 + 2\pi\tilde{r}_A T_{\text{H}}) dF = -\left( \frac{\hat{E}}{T_{\text{H}}} + \hat{S} \right) \frac{dF}{F} \quad (3.24)$$

$$= \frac{6\pi}{G} \frac{8HT + \dot{T}}{T(4HT + \dot{T})} dF. \quad (3.25)$$

The additional term  $d_i \hat{S}$  in Eq. (3.23) can be interpreted as an entropy production term in the non-equilibrium thermodynamics. In  $f(T)$  gravity,  $d_i \hat{S}$  in Eq. (3.24) is non-vanishing due to  $dF \neq 0$  unless  $f(T) = T$  which gives rise to  $F = 1$  and  $d_i \hat{S} = 0$ . As a result, the first-law of equilibrium thermodynamics holds.

### C. Second law of thermodynamics

Normally, in the cosmological FLRW fluid dynamics the entropy is simply the fluid-entropy current and has little to do with the horizon entropy. In the flat FLRW space-time, the Bekenstein-Hawking horizon entropy of the apparent horizon is expressed as  $S = A/(4G) = \pi/(GH^2) \propto H^{-2}$ , where in deriving the first equality we have used  $A = 4\pi\tilde{r}_A^2$  and Eq. (3.6). On the other hand, for modified gravitational theories including  $f(R)$  and  $f(T)$ , there can exist the phantom phase in which  $\dot{H} > 0$ . In such phase,  $\dot{S} = -2[\pi/(GH^3)] \dot{H} < 0$  and hence, the second law of thermodynamics in terms of the horizon entropy is not satisfied. Thus, this seems to imply that such modified gravitational theories with the phantom phase cannot be an alternative gravitational theory to general relativity. However, it is not the



case. If we investigate the entropy of the total energy of the horizon, i.e., the sum of the horizon entropy and the entropy of ordinary perfect fluids of generic matter, the total entropy always increases in time and the second law of thermodynamics can be met, as explicitly shown in  $f(R)$  gravity in Ref. [31]. In this subsection, we examine this point in  $f(T)$  gravity.

To explore the second law of thermodynamics in  $f(T)$  gravity, we start with the Gibbs equation in terms of all matter and energy fluid, given by

$$T_{\text{H}} d\hat{S}_{\text{t}} = d(\hat{\rho}_{\text{t}} V) + \hat{P}_{\text{t}} dV = V d\hat{\rho}_{\text{t}} + (\hat{\rho}_{\text{t}} + \hat{P}_{\text{t}}) dV, \quad (3.26)$$

where  $T_{\text{H}}$  and  $\hat{S}_{\text{t}}$  denote the temperature and entropy of total energy inside the horizon, respectively. Here, we have assumed the same temperature between the outside and inside of the apparent horizon. To obey the second law of thermodynamics in  $f(T)$  gravity, we require that [43]

$$\Xi \equiv \frac{d\hat{S}}{dt} + \frac{d(d_i \hat{S})}{dt} + \frac{d\hat{S}_{\text{t}}}{dt} \geq 0. \quad (3.27)$$

By using Eqs. (3.1), (3.23) and (3.26), we obtain

$$\Xi = -\frac{3}{4G} \frac{\dot{T}^2 F}{T^3}. \quad (3.28)$$

Since  $-T^3 = 216H^6 > 0$ , we see that the condition of  $\Xi \geq 0$  becomes

$$J \equiv \dot{T}^2 F = 144H^2 \dot{H}^2 F \geq 0, \quad (3.29)$$

which is always met because  $F > 0$  in order that  $\hat{E} > 0$ . Hence, the second law of thermodynamics in  $f(T)$  gravity can be satisfied. It is clear from Eq. (3.29) that  $J \geq 0$  irrespective of the sign of  $\dot{H}$ . This result is also consistent with a phantom model with ordinary thermodynamics [48]. Investigations on entropy in phantom models have also been executed [49].

We remark that we have used the physical temperature as the temperature of the apparent horizon, i.e., the Hawking temperature, given by  $T_{\text{H}} = |\kappa_{\text{sg}}|/(2\pi)$  in Eq. (3.11). This temperature clearly depends on the energy momentum tensor of the dark components coming from  $f(T)$  gravity as demonstrated in, e.g., Eqs. (3.12) and (3.13). On the other hand, in a cosmological setup the temperature of matter species, such as about 2.73K of the CMB photons is determined in a standard way. In our discussions on the second law of thermodynamics in  $f(T)$  gravity, we have concentrated on the case in which the temperature of the universe inside the horizon is equal to that of the apparent horizon. In other words, the temperature of the apparent horizon is assumed to be the same as the temperature of matter species including that of the CMB photons.

#### IV. EQUILIBRIUM DESCRIPTION OF THERMODYNAMICS IN $f(T)$ GRAVITY

Since there is a non-equilibrium entropy production term  $d_i\hat{S}$ , the right-hand side (r.h.s.) of Eq. (3.5) does not vanish, i.e., the standard continuity equation for  $\hat{\rho}_{\text{DE}}$  and  $\hat{P}_{\text{DE}}$  defined in Eqs. (3.3) and (3.4) does not hold. In this section, it is demonstrated that by redefining the energy density and pressure of dark components to meet the continuity equation, there can be no extra entropy production term. We refer to this as the equilibrium description in  $f(T)$  gravity.

##### A. Energy density and pressure of dark components

By comparing the gravitational field equations (2.9) and (2.10) with the ordinary ones in general relativity:

$$H^2 = \frac{\kappa^2}{3} (\rho_{\text{M}} + \rho_{\text{DE}}) , \quad (4.1)$$

$$\dot{H} = -\frac{\kappa^2}{2} (\rho_{\text{M}} + P_{\text{M}} + \rho_{\text{DE}} + P_{\text{DE}}) , \quad (4.2)$$

the energy density and pressure of dark components can be rewritten as

$$\rho_{\text{DE}} = \frac{1}{2\kappa^2} (-T - f + 2TF) , \quad (4.3)$$

$$P_{\text{DE}} = -\frac{1}{2\kappa^2} \left[ 4(1 - F - 2TF') \dot{H} + (-T - f + 2TF) \right] , \quad (4.4)$$

which clearly satisfy the standard continuity equation, i.e.,

$$\dot{\rho}_{\text{DE}} + 3H(\rho_{\text{DE}} + P_{\text{DE}}) = 0 . \quad (4.5)$$

##### B. First law of thermodynamics

In the representation of Eqs. (4.1) and (4.2), Eq. (3.8) becomes

$$\frac{1}{4\pi G} d\tilde{r}_A = \tilde{r}_A^3 H (\rho_{\text{t}} + P_{\text{t}}) dt , \quad (4.6)$$

with  $\rho_{\text{t}} \equiv \rho_{\text{DE}} + \rho_{\text{M}}$  and  $P_{\text{t}} \equiv P_{\text{DE}} + P_{\text{M}}$ . By introducing the horizon entropy  $S = A/(4G)$  and using Eq. (4.6), we have

$$\frac{1}{2\pi\tilde{r}_A} dS = 4\pi\tilde{r}_A^3 H (\rho_{\text{t}} + P_{\text{t}}) dt . \quad (4.7)$$

From the horizon temperature in Eq. (3.14) and Eq. (4.7), we find

$$T_H dS = 4\pi \tilde{r}_A^3 H (\rho_t + P_t) dt - 2\pi \tilde{r}_A^2 (\rho_t + P_t) d\tilde{r}_A. \quad (4.8)$$

By defining the Misner-Sharp energy as

$$E = \frac{\tilde{r}_A}{2G} = V \rho_t, \quad (4.9)$$

we get

$$dE = -4\pi \tilde{r}_A^3 H (\rho_t + P_t) dt + 4\pi \tilde{r}_A^2 \rho_t d\tilde{r}_A, \quad (4.10)$$

where there does not exist any additional term proportional to  $dF$  on the r.h.s. due to the continuity equation (4.6). From Eqs. (4.8) and (4.10), we obtain the following equation corresponding to the first law of equilibrium thermodynamics:

$$T_H dS = -dE + W dV \quad (4.11)$$

with the work density  $W$  given by

$$W = \frac{1}{2} (\rho_t - P_t). \quad (4.12)$$

Thus, by redefining  $\rho_{DE}$  and  $P_{DE}$  so that the continuity equation (4.5) can be met, we can realize an equilibrium description of thermodynamics.

Furthermore, it follows from Eqs. (4.1), (4.2) and (4.7) that

$$\dot{S} = 8\pi^2 H \tilde{r}_A^4 (\rho_t + P_t) = \frac{6\pi}{G} \frac{\dot{T}}{T^2}. \quad (4.13)$$

Since  $\dot{S} \propto \dot{T}/T^2 \propto -\dot{H}/H^3$ , the horizon entropy increases in the expanding universe as long as the null energy condition  $\rho_t + P_t \geq 0$  is satisfied, in which  $\dot{H} \leq 0$ .

There are two main reasons why we can obtain the equilibrium description of thermodynamics. One is that we can redefine the energy density and pressure of dark components so that the standard continuity equation can be satisfied. The other is that the horizon entropy  $S$  is proportional to the horizon area  $A$  in  $S = A/(4G)$  as general relativity<sup>3</sup>.

The relation between the horizon entropy  $S$  in the equilibrium description and  $\hat{S}$  in the non-equilibrium description can be described as [30]

$$dS = d\hat{S} + d_i \hat{S} + \frac{\tilde{r}_A}{2GT_H} dF - \frac{2\pi(1-F)}{G} \frac{\dot{H}}{H^3} dt. \quad (4.14)$$

---

<sup>3</sup> There also exist studies on the expression of the horizon entropy in the four-dimensional modified gravity [50] and the quantum logarithmic correction to it in cosmological settings [51].

By using the relations (3.25) and (4.7), Eq. (4.14) is rewritten to the following form:

$$dS = \frac{1}{F}d\hat{S} + \frac{1}{F} \frac{2H^2 + \dot{H}}{4H^2 + \dot{H}} d_i\hat{S}, \quad (4.15)$$

where  $d_i\hat{S}$  is given by Eq. (3.25). The difference between  $S$  and  $\hat{S}$  appears in  $f(T)$  gravity due to  $dF \neq 0$ , whereas  $S = \hat{S}$  in the theory of  $f(T) = T$  because  $F = 1$ . Eq. (4.15) implies that  $dS$  in the equilibrium framework includes the information of  $d\hat{S}$  as well as  $d_i\hat{S}$  in the non-equilibrium framework.

In the flat FLRW space-time, since the Bekenstein-Hawking entropy behaves as  $S \propto H^{-2}$  regardless of gravity theories,  $S$  increases if  $H$  decreases, whereas when  $H$  becomes large,  $S$  becomes small (similar to that in the case of superinflation), i.e.,  $S$  increases and decreases for  $w_t \equiv P_t/\rho_t > -1$  ( $\dot{H} < 0$ ) and  $w_t < -1$  ( $\dot{H} > 0$ ), respectively. That is, such a property is equivalent to the standard general relativistic picture with the energy density  $\rho_{\text{DE}}$  in Eq. (4.3) and the pressure  $P_{\text{DE}}$  in Eq. (4.4) of dark components.

On the other hand, the Wald entropy evolves as  $\hat{S} \propto FH^{-2}$  with involving the information of theories. For instance, in a model of  $f(T) = T + \alpha T^n$ , where  $\alpha$  and  $n$  are constants,  $\hat{S} \propto H^{2(n-2)}$  and hence  $\hat{S}$  becomes large apart from  $n = 2$  because  $H$  increases (decreases) for  $n > 2$  ( $n < 2$ ). Therefore, the behavior of the Wald entropy is different from that of the Bekenstein-Hawking entropy. The entropy production term  $d_i\hat{S}$  in the non-equilibrium description presents a relation to the equilibrium description as given in Eq. (4.15), i.e.,  $dS$  consists of  $d\hat{S}$  and  $d_i\hat{S}$ . As a result, the equilibrium framework is more transparent than the non-equilibrium one because it gives not only the general relativistic analogue of the horizon entropy independent of gravitational theories but also the more profound connection of the non-equilibrium thermodynamics with the standard equilibrium one.

It is interesting to mention that there exists the difference between the EoS of dark components in the two descriptions. From Eqs. (3.3), (3.4), (4.3) and (4.4), we find that

$$\hat{w}_{\text{DE}} = \frac{\hat{P}_{\text{DE}}}{\hat{\rho}_{\text{DE}}} = -1 + \frac{4H\dot{F}}{FT - f}, \quad (4.16)$$

$$w_{\text{DE}} = \frac{P_{\text{DE}}}{\rho_{\text{DE}}} = -1 - \frac{4(1 - F - 2TF')\dot{H}}{-T - f + 2TF}, \quad (4.17)$$

in the non-equilibrium and equilibrium descriptions, respectively. It is easy to see that in general  $\hat{w}_{\text{DE}} \neq w_{\text{DE}}$  except the theory of  $f(T) = T$ . Consequently, in order to compare the theoretical prediction in terms of the EoS of dark components in  $f(T)$  gravity with

the observations, the expression of the EoS of dark components in the non-equilibrium description as well as that in the equilibrium one are necessary. This is a meaningful physical implication of the results from the physics perspective obtained by exploring both non-equilibrium and equilibrium descriptions.

### C. Second law of thermodynamics

To examine the second law of thermodynamics in the equilibrium description, we write the Gibbs equation in terms of all matter and energy fluid as

$$T_H dS_t = d(\rho_t V) + P_t dV = V d\rho_t + (\rho_t + P_t) dV. \quad (4.18)$$

The second law of thermodynamics can be described by

$$\frac{dS_{\text{sum}}}{dt} \equiv \frac{dS}{dt} + \frac{dS_t}{dt} \geq 0, \quad (4.19)$$

where  $S_{\text{sum}} \equiv S + S_t$ . Consequently, we obtain

$$\frac{dS_{\text{sum}}}{dt} = -\frac{6\pi}{G} \left( \frac{\dot{T}}{T} \right)^2 \frac{1}{4HT + \dot{T}}, \quad (4.20)$$

by using  $V = 4\pi\tilde{r}_A^3/3$ , and Eqs. (3.14), (4.2) and (4.13). Hence, the relation (4.19) with Eq. (4.20) leads to the condition

$$Y \equiv -\left(4HT + \dot{T}\right) = 12H\left(2H^2 + \dot{H}\right) \geq 0. \quad (4.21)$$

In the flat FLRW expanding background ( $H > 0$ ), the second law of thermodynamics can be satisfied as long as the quantity  $\left(2H^2 + \dot{H}\right)$  is positive or equal to zero. It is interesting to note that in  $f(R)$  gravity,  $R = 6\left(2H^2 + \dot{H}\right)$  [31] for the flat FLRW space-time and the condition in Eq. (4.21) clearly holds as  $R \geq 0$ . By analogy with  $f(R)$  gravity, we can extend the condition, i.e.,  $Y \geq 0$ , to  $f(T)$  gravity since  $Y$  involves only  $H$  and  $\dot{H}$  and is related to the scalar curvature in general relativity. Thus, we have acquired a unified insight between non-equilibrium and equilibrium descriptions of thermodynamics. It should be cautioned that this result can be shown explicitly only for the same temperature of the universe outside and inside the apparent horizon [52].

## V. CONCLUSIONS

We have investigated the first and second laws of thermodynamics of the apparent horizon in  $f(T)$  gravity with both non-equilibrium and equilibrium descriptions. We have found the same dual equilibrium/non-equilibrium formulation for  $f(T)$  as for  $f(R)$  gravity. We have demonstrated that the second law of thermodynamics can be satisfied in the non-equilibrium and equilibrium frameworks if the temperature of the universe inside the horizon is equal to that of the apparent horizon. It has been shown that in the non-equilibrium framework, the second law of thermodynamics can be met regardless of the sign of the time derivative of the Hubble parameter, whereas in the equilibrium framework, the second law of thermodynamics can be verified by analogy with the same non-negative quantity related to the scalar curvature in general relativity is positive or equal to zero in the expanding cosmological background.

Finally, we emphasize that our result of the second law of thermodynamics in  $f(T)$  gravity is nontrivial and meaningful. We believe that any successful alternative gravitational theory to general relativity should obey this law. If the law is violated in certain universes in a model, it is more likely to be due to an incorrect generalization of the second law or some inherent inconsistency of the model itself. For the latter, the model should be abandoned. Furthermore, it is important to note that the same temperature inside and on the apparent horizon is a working hypothesis as it may not be so generally.

## ACKNOWLEDGMENTS

We would like to thank Professor Shinji Tsujikawa for his collaboration in our previous work [30]. The work is supported in part by the National Science Council of R.O.C. under Grant #: NSC-98-2112-M-007-008-MY3 and National Tsing Hua University under the Boost Program #: 99N2539E1.

---

[1] S. Perlmutter *et al.* [SNCP Collaboration], *Astrophys. J.* **517**, 565 (1999); A. G. Riess *et al.* [SNST Collaboration], *Astron. J.* **116**, 1009 (1998).

- [2] D. N. Spergel *et al.* [WMAP Collaboration], *Astrophys. J. Suppl.* **148**, 175 (2003); *ibid.* **170**, 377 (2007); E. Komatsu *et al.* [WMAP Collaboration], *ibid.* **180**, 330 (2009).
- [3] E. Komatsu *et al.* [WMAP Collaboration], *Astrophys. J. Suppl.* **192**, 18 (2011).
- [4] M. Tegmark *et al.*, *Phys. Rev. D* **69**, 103501 (2004); U. Seljak *et al.* [SDSS Collaboration], *Phys. Rev. D* **71**, 103515 (2005).
- [5] D. J. Eisenstein *et al.*, *Astrophys. J.* **633**, 560 (2005).
- [6] B. Jain and A. Taylor, *Phys. Rev. Lett.* **91**, 141302 (2003).
- [7] E. J. Copeland, M. Sami and S. Tsujikawa, *Int. J. Mod. Phys. D* **15**, 1753 (2006).
- [8] M. Li, X. D. Li, S. Wang and Y. Wang, *Commun. Theor. Phys.* **56**, 525 (2011) [arXiv:1103.5870 [astro-ph.CO]].
- [9] S. Nojiri and S. D. Odintsov, *Phys. Rept.* **505**, 59 (2011); eConf **C0602061**, 06 (2006) [Int. J. Geom. Meth. Mod. Phys. **4**, 115 (2007)] [arXiv:hep-th/0601213].
- [10] T. P. Sotiriou and V. Faraoni, *Rev. Mod. Phys.* **82**, 451 (2010).
- [11] A. De Felice and S. Tsujikawa, *Living Rev. Rel.* **13**, 3 (2010).
- [12] T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, arXiv:1106.2476 [astro-ph.CO].
- [13] F. W. Hehl, P. Von Der Heyde, G. D. Kerlick and J. M. Nester, *Rev. Mod. Phys.* **48**, 393 (1976); K. Hayashi and T. Shirafuji, *Phys. Rev. D* **19**, 3524 (1979) [Addendum-*ibid.* **D 24**, 3312 (1982)]; E. E. Flanagan and E. Rosenthal, *ibid.* **75**, 124016 (2007); J. Garecki, arXiv:1010.2654 [gr-qc].
- [14] A. Einstein, *Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl.*, 217 (1928); 401 (1930); A. Einstein, *Math. Ann.* **102**, 685 (1930).
- [15] G. R. Bengochea and R. Ferraro, *Phys. Rev. D* **79**, 124019 (2009).
- [16] E. V. Linder, *Phys. Rev. D* **81**, 127301 (2010) [Erratum-*ibid.* **D 82**, 109902 (2010)].
- [17] R. Ferraro and F. Fiorini, *Phys. Rev. D* **75**, 084031 (2007); *ibid.* **78**, 124019 (2008).
- [18] P. Wu and H. W. Yu, *Phys. Lett. B* **693**, 415 (2010); R. Myrzakulov, arXiv:1006.1120 [gr-qc]; K. K. Yerzhanov, S. R. Myrzakul, I. I. Kulnazarov and R. Myrzakulov, arXiv:1006.3879 [gr-qc]; P. Wu and H. Yu, *Phys. Lett. B* **692**, 176 (2010); R. J. Yang, arXiv:1007.3571 [gr-qc]; P. Y. Tsyba, I. I. Kulnazarov, K. K. Yerzhanov and R. Myrzakulov, *Int. J. Theor. Phys.* **50**, 1876 (2011); S. H. Chen, J. B. Dent, S. Dutta and E. N. Saridakis, *Phys. Rev. D* **83**, 023508 (2011); G. R. Bengochea, *Phys. Lett. B* **695**, 405 (2011); P. Wu and H. W. Yu, *Eur. Phys. J. C* **71**, 1552 (2011); R. Myrzakulov, arXiv:1008.4486 [astro-ph.CO]; K. Karami and A. Abdol-

- maleki, arXiv:1009.2459 [gr-qc]; arXiv:1009.3587 [physics.gen-ph]; R. J. Yang, Europhys. Lett. **93**, 60001 (2011); J. B. Dent, S. Dutta and E. N. Saridakis, JCAP **1101**, 009 (2011); T. Wang, Phys. Rev. D **84**, 024042 (2011); Y. Zhang, H. Li, Y. Gong and Z. H. Zhu, JCAP **1107**, 015 (2011); C. Deliduman and B. Yapiskan, arXiv:1103.2225 [gr-qc]; B. Li, T. P. Sotiriou and J. D. Barrow, arXiv:1103.2786 [astro-ph.CO]; B. Li, T. P. Sotiriou and J. D. Barrow, Phys. Rev. D **83**, 104017 (2011); Y. F. Cai, S. H. Chen, J. B. Dent, S. Dutta and E. N. Saridakis, arXiv:1104.4349 [astro-ph.CO]; S. Chattopadhyay and U. Debnath, Int. J. Mod. Phys. D **20**, 1135 (2011); P. B. Khatua, S. Chakraborty and U. Debnath, arXiv:1105.3393 [physics.gen-ph]; M. Sharif and S. Rani, Mod. Phys. Lett. A **26**, 1657 (2011); H. Wei, X. P. Ma and H. Y. Qi, Phys. Lett. B **703**, 74 (2011); X. h. Meng and Y. b. Wang, arXiv:1107.0629 [astro-ph.CO]; C. G. Boehmer, A. Mussa and N. Tamanini, arXiv:1107.4455 [gr-qc]; H. Wei, H. Y. Qi and X. P. Ma, arXiv:1108.0859 [gr-qc]; S. Capozziello, V. F. Cardone, H. Farajollahi and A. Ravanpak, arXiv:1108.2789 [astro-ph.CO]; M. H. Daouda, M. E. Rodrigues and M. J. S. Houndjo, arXiv:1108.2920 [astro-ph.CO]; L. R. A. Belo, E. P. Spaniol, J. A. de Deus and V. C. de Andrade, arXiv:1108.3796 [gr-qc]; P. Wu and H. Yu, Phys. Lett. B **703**, 223 (2011); M. H. Daouda, M. E. Rodrigues and M. J. S. Houndjo, arXiv:1109.0528 [gr-qc]; R. Ferraro and F. Fiorini, arXiv:1109.4209 [gr-qc]; C. Q. Geng, C. C. Lee, E. N. Saridakis, Y. P. Wu, Phys. Lett. **B704**, 384 (2011) [arXiv:1109.1092 [hep-th]]; H. Wei, arXiv:1109.6107 [gr-qc]; C. Q. Geng, C. C. Lee and E. N. Saridakis, arXiv:1110.0913 [astro-ph.CO]; P. A. Gonzalez, E. N. Saridakis, Y. Vasquez, arXiv:1110.4024 [gr-qc].
- [19] K. Bamba, C. Q. Geng and C. C. Lee, arXiv:1008.4036 [astro-ph.CO]; see also C. Q. Geng, talk presented at the 2nd Chuang-Xin group meeting on “Dark Energy and Dark Matter” , Weihai, China, July 19–Aug. 3, 2010; K. Bamba, C. Q. Geng, C. C. Lee and L. W. Luo, JCAP **1101**, 021 (2011) [arXiv:1011.0508 [astro-ph.CO]].
- [20] R. Zheng and Q. G. Huang, JCAP **1103**, 002 (2011) [arXiv:1010.3512 [gr-qc]].
- [21] B. Li, T. P. Sotiriou and J. D. Barrow, Phys. Rev. D **83**, 064035 (2011); T. P. Sotiriou, B. Li and J. D. Barrow, *ibid.* **83**, 104030 (2011); R. Ferraro and F. Fiorini, Phys. Lett. B **702**, 75 (2011); M. Li, R. X. Miao and Y. G. Miao, JHEP **1107**, 108 (2011).
- [22] R. X. Miao, M. Li and Y. G. Miao, arXiv:1107.0515 [hep-th].
- [23] J. M. Bardeen, B. Carter and S. W. Hawking, Commun. Math. Phys. **31**, 161 (1973); J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973); S. W. Hawking, Commun. Math. Phys. **43**,



- 199 (1975) [Erratum-ibid. **46**, 206 (1976)]; G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2738 (1977).
- [24] T. Padmanabhan, arXiv:0910.0839 [gr-qc]; AIP Conf. Proc. **1241**, 93 (2010) [arXiv:0911.1403 [gr-qc]]; Rept. Prog. Phys. **73**, 046901 (2010).
- [25] S. Kolekar and T. Padmanabhan, Phys. Rev. D **82**, 024036 (2010).
- [26] T. Jacobson, Phys. Rev. Lett. **75**, 1260 (1995).
- [27] E. Elizalde and P. J. Silva, Phys. Rev. D **78**, 061501 (2008); K. Bamba, C. Q. Geng, S. Nojiri and S. D. Odintsov, Europhys. Lett. **89**, 50003 (2010); S. F. Wu, B. Wang, X. H. Ge and G. H. Yang, Phys. Rev. D **81**, 044010 (2010); Y. Yokokura, arXiv:1106.3149 [hep-th].
- [28] R. Brustein and M. Hadad, Phys. Rev. Lett. **103**, 101301 (2009).
- [29] C. Eling, R. Guedens and T. Jacobson, Phys. Rev. Lett. **96**, 121301 (2006).
- [30] K. Bamba, C. Q. Geng and S. Tsujikawa, Phys. Lett. B **688**, 101 (2010) [arXiv:0909.2159 [gr-qc]]; Int. J. Mod. Phys. D **20**, 1363 (2011) [arXiv:1101.3628 [gr-qc]].
- [31] K. Bamba and C. Q. Geng, JCAP **1006**, 014 (2010) [arXiv:1005.5234 [gr-qc]].
- [32] J. Zhou, B. Wang, Y. Gong and E. Abdalla, Phys. Lett. B **652**, 86 (2007).
- [33] B. Wang, Y. Gong and E. Abdalla, Phys. Rev. D **74**, 083520 (2006).
- [34] R. M. Wald, Phys. Rev. D **48**, 3427 (1993); V. Iyer and R. M. Wald, *ibid.* **50**, 846 (1994).
- [35] R. Brustein, D. Gorbonos and M. Hadad, Phys. Rev. D **79**, 044025 (2009).
- [36] T. Jacobson, G. Kang and R. C. Myers, Phys. Rev. D **49**, 6587 (1994); G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini, JCAP **0502**, 010 (2005).
- [37] D. N. Vollick, Phys. Rev. D **76**, 124001 (2007).
- [38] W. Gu, M. Li and R. X. Miao, arXiv:1011.3419 [hep-th]; R. X. Miao, J. Meng and M. Li, arXiv:1102.1166 [hep-th].
- [39] Y. P. Wu and C. Q. Geng, arXiv:1110.3099 [gr-qc].
- [40] R. G. Cai and S. P. Kim, JHEP **0502**, 050 (2005).
- [41] C. W. Misner and D. H. Sharp, Phys. Rev. **136**, B571 (1964); D. Bak and S. J. Rey, Class. Quant. Grav. **17**, L83 (2000).
- [42] S. F. Wu, B. Wang and G. H. Yang, Nucl. Phys. B **799**, 330 (2008).
- [43] S. F. Wu, B. Wang, G. H. Yang and P. M. Zhang, Class. Quant. Grav. **25**, 235018 (2008).
- [44] Y. Gong and A. Wang, Phys. Rev. Lett. **99**, 211301 (2007).

- [45] N. Sakai and J. D. Barrow, *Class. Quant. Grav.* **18**, 4717 (2001); R. G. Cai, L. M. Cao, Y. P. Hu and N. Ohta, *Phys. Rev. D* **80**, 104016 (2009).
- [46] A. A. Starobinsky, *JETP Lett.* **86**, 157 (2007).
- [47] S. A. Hayward, *Class. Quant. Grav.* **15**, 3147 (1998); S. A. Hayward, S. Mukohyama and M. C. Ashworth, *Phys. Lett. A* **256**, 347 (1999); R. G. Cai and L. M. Cao, *Phys. Rev. D* **75**, 064008 (2007).
- [48] S. Nojiri and S. D. Odintsov, *Phys. Rev. D* **72**, 023003 (2005).
- [49] I. H. Brevik, S. Nojiri, S. D. Odintsov and L. Vanzo, *Phys. Rev. D* **70**, 043520 (2004); S. Nojiri and S. D. Odintsov, *ibid.* **70**, 103522 (2004); N. Bilic, *ibid.* **78**, 105012 (2008); K. Bamba and C. Q. Geng, *Phys. Lett. B* **679**, 282 (2009) [arXiv:0901.1509 [hep-th]].
- [50] P. Wang, *Phys. Rev. D* **72**, 024030 (2005).
- [51] R. G. Cai, L. M. Cao and Y. P. Hu, *JHEP* **0808**, 090 (2008); T. Zhu, J. R. Ren and M. F. Li, *Phys. Lett. B* **674**, 204 (2009); J. E. Lidsey, *Class. Quant. Grav.* **26**, 147001 (2009); arXiv:0911.3286 [hep-th]; R. G. Cai, L. M. Cao and N. Ohta, *JHEP* **1004**, 082 (2010); H. Mohseni Sadjadi and M. Jamil, *Europhys. Lett.* **92**, 69001 (2010).
- [52] Y. Gong, B. Wang and A. Wang, *JCAP* **0701**, 024 (2007); M. Jamil, E. N. Saridakis and M. R. Setare, *Phys. Rev. D* **81**, 023007 (2010).